NETS OF SPACE CURVES*

BY

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In the metric differential geometry of a curved surface, the discussion is often centered about the gaussian parameter form of representation, in which the points on the surface are fixed by an arbitrary system of curvilinear coördinates. So far as we are aware, the configuration of two arbitrary families of curves on a surface, i.e., an arbitrary net or system of curvilinear coördinates, has not been studied for its own sake. By this we mean that the properties usually investigated belong to the surface, and not intrinsically to the net itself.

It is our purpose in the present paper to enter upon this apparently neglected field, by studying projectively an arbitrary net of space curves. The point of departure is a completely integrable system of partial differential equations, any fundamental system of solutions of which defines the geometric configuration, and which has moreover the property that the similar configuration defined by any other fundamental system of solutions is a projective transformation of the first.† The projective properties of the net will therefore be expressed in terms of the differential equations. Of course all properties belonging to the surface on which the net lies belong also to the net itself, but we shall find that there are in addition many others peculiar to the net.

1. The differential equations of the problem

Suppose the homogeneous coördinates $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ of a point in space to be given as analytic functions of two variables:

* Presented to the Society, April 24, 1915.

The manuscript of this paper was found among the papers in the late Dr. Green's handwriting which were turned over to me for investigation. It was complete except for a few references, which I have supplied; the only changes which I have permitted myself to make are concerned with two or three places where the language seemed to be ambiguous. This paper constitutes a notable addition to Green's writings, and shows once more the fertility of his imagination and his unfailing ability to see something new even in the most familiar fields.—E. J. Wilczynski.

† The first to use systematically completely integrable systems of this kind in projective differential geometry was Wilczynski. See for instance his five memoirs on the theory of surfaces, these Transactions (1907–1910).

$$(1) \qquad y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4).$$

For $v =$ const., these equations define a one-parameter family of curves C_u, and for $u =$ const., a one-parameter family of curves C_v; the net formed by the curves C_u and C_v lies on a surface S_y.

In our subsequent analysis we shall suppose that the curves C_u are not conjugate to the curves C_v. In other words, the four functions (1) are not all solutions of an equation of the Laplace type:*

$$(2) \qquad \alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0.$$

We may express this otherwise, by saying that the determinant

$$(3) \qquad W = \begin{vmatrix} y_{uv}^{(1)} & y_u^{(1)} & y_v^{(1)} & y^{(1)} \\ y_{uv}^{(2)} & y_u^{(2)} & y_v^{(2)} & y^{(2)} \\ y_{uv}^{(3)} & y_u^{(3)} & y_v^{(3)} & y^{(3)} \\ y_{uv}^{(4)} & y_u^{(4)} & y_v^{(4)} & y^{(4)} \end{vmatrix}$$

is not zero. Geometrically, this means that for values of u and v corresponding to points in the region of space for which $W \neq 0$, the four points $y_{uv}^{(k)}$, $y_u^{(k)}$, $y_v^{(k)}$, $y^{(k)}$, $(k = 1, 2, 3, 4)$ are not coplanar, so that any other point of space must be linearly dependent on these four. In particular, this must be the case with the two points $y_{uu}^{(k)}$ and $y_{vv}^{(k)}$, so that relations of the form

$$(4) \qquad \begin{aligned} y_{uu}^{(k)} &= a y_{uv}^{(k)} + b y_u^{(k)} + c y_v^{(k)} + d y^{(k)}, \\ y_{vv}^{(k)} &= a' y_{uv}^{(k)} + b' y_u^{(k)} + c' y_v^{(k)} + d' y^{(k)} \end{aligned} \qquad (k = 1, 2, 3, 4)$$

must exist, where the coefficients a, b, etc., are functions of u and v. *The functions* (1) *are solutions of the partial differential equations*

$$(5) \qquad \begin{aligned} y_{uu} &= a y_{uv} + b y_u + c y_v + d y, \\ y_{vv} &= a' y_{uv} + b' y_u + c' y_v + d' y. \end{aligned}$$

In fact, we may determine the coefficients in these differential equations by substituting successively $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ for y, thus obtaining the eight equations (4), and then solving these for a, b, etc. This is possible, since we have supposed the determinant W given by (3) to be different from zero.

Similarly, all derivatives of order higher than the second are, like y_{uu} and y_{vv}, expressible linearly in terms of y_{uv}, y_u, y_v, y; in fact, these expressions

* Darboux, *Théorie générale des surfaces*, vol. 1, p. 122. Conjugate nets, as is well known, have received a good deal of attention, especially because of their intimate relation to the theory of congruences. For a discussion of the subject from the point of view of the present paper, see G. M. Green, *Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface*, (First Memoir), American Journal of Mathematics, vol. 37 (1915), pp. 215–246.

may actually be found by repeated differentiation of equations (5). It is apparent, then, that system (5) is completely integrable, and has the functions $y^{(k)}$ $(k = 1, 2, 3, 4)$ as a fundamental set of solutions. Moreover, any other fundamental set of solutions $\bar{y}^{(k)}$—i.e., a set for which, as for $y^{(k)}$, the determinant $|\bar{y}_{uv}, \bar{y}_u, \bar{y}_v, \bar{y}|$ is different from zero—is expressible linearly, with constant coefficients, in terms of the original set $y^{(k)}$:

$$\bar{y}^{(k)} = c_{k1} y^{(1)} + c_{k2} y^{(2)} + c_{k3} y^{(3)} + c_{k4} y^{(4)}, \qquad |c_{kl}| \neq 0 \quad (k = 1, 2, 3, 4). \tag{6}$$

In other words, *the net defined by any fundamental system of solutions of equations* (5) *is a projective transformation of the net defined by any other fundamental system.* Consequently, any geometric property of the net, expressible in terms of the differential equations, is common to the net and all of its projective transformations; in other words, the property is projective.

We shall base our study entirely upon the system of differential equations; evidently, if a system of the form of equations (5) be written down at random, it will be incompatible, so that relations must exist among the coefficients in order that the system be completely integrable and have the properties just described. We consider therefore this question. By differentiation of equations (5) we obtain the four equations

$$\begin{aligned}
y_{uuu} - a y_{uuv} &= (a_u + ab + c) y_{uv} + (b^2 + b_u + d) y_u \\
&\qquad + (bc + c_u) y_v + (bd + d_u) y, \\
y_{uuv} - a y_{uvv} &= (a_v + b + ca') y_{uv} + (b_v + cb') y_u \\
&\qquad + (cc' + c_v + d) y_v + (cd' + d_v) y, \\
y_{uvv} - a' y_{uuv} &= (a'_u + b' a + c') y_{uv} + (b' b + b'_u + d') y_u \\
&\qquad + (b' c + c'_u) y_v + (b' d + d'_u) y, \\
y_{vvv} - a' y_{uvv} &= (a'_v + b' + c' a') y_{uv} + (b'_v + c' b') y_u \\
&\qquad + (c'^2 + c'_v + d') y_v + (c' d' + d'_v) y,
\end{aligned} \tag{7}$$

in which use has been made of equations (5) to replace y_{uu} and y_{vv} by their values in terms of y_{uv}, y_u, y_v, y. From the second and third of equations (7) may be obtained the values for the derivatives y_{uuv} and y_{uvv}, and then these may be used to find y_{uuu} and y_{vvv} from the first and fourth of equations (7). In this way we obtain for the four derivatives of the third order the expressions

$$\begin{aligned}
y_{uuu} &= a^{(30)} y_{uv} + b^{(30)} y_u + c^{(30)} y_v + d^{(30)} y, \\
y_{uuv} &= a^{(21)} y_{uv} + b^{(21)} y_u + c^{(21)} y_v + d^{(21)} y, \\
y_{uvv} &= a^{(12)} y_{uv} + b^{(12)} y_u + c^{(12)} y_v + d^{(12)} y, \\
y_{vvv} &= a^{(03)} y_{uv} + b^{(03)} y_u + c^{(03)} y_v + d^{(03)} y,
\end{aligned} \tag{8}$$

where

$$
(9)\qquad
\begin{aligned}
a^{(21)} &= [a_v + b + ca' + a(a'_u + b'a + c')]/(1 - aa'),\\
b^{(21)} &= [b_v + cb' + a(b'b + b'_u + d')]/(1 - aa'),\\
c^{(21)} &= [cc' + c_v + d + a(b'c + c'_u)]/(1 - aa'),\\
d^{(21)} &= [cd' + d_v + a(b'd + d'_u)]/(1 - aa'),\\
a^{(12)} &= [a'(a_v + b + ca') + a'_u + b'a + c']/(1 - aa'),\\
b^{(12)} &= [a'(b_v + cb') + b'b + b'_u + d']/(1 - aa'),\\
c^{(12)} &= [a'(cc' + c_v + d) + b'c + c'_u]/(1 - aa'),\\
d^{(12)} &= [a'(cd' + d_v) + b'd + d'_u]/(1 - aa'),\\
a^{(30)} &= aa^{(21)} + a_u + ab + c, \qquad & b^{(30)} &= ab^{(21)} + b^2 + b_u + d,\\
c^{(30)} &= ac^{(21)} + bc + c_u, & d^{(30)} &= ad^{(21)} + bd + d_u,\\
a^{(03)} &= a'a^{(12)} + a'_v + b' + c'a', & b^{(03)} &= a'b^{(12)} + b'_v + c'b',\\
c^{(03)} &= a'c^{(12)} + c'^2 + c'_v + d', & d^{(03)} &= a'd^{(12)} + c'd' + d'_v.
\end{aligned}
$$

We suppose, of course, that

$$(10)\qquad 1 - aa' \neq 0,$$

which is tantamount to assuming that the surface is not developable.*

The derivatives of the fourth order are likewise expressible linearly in terms of y_{uv}, y_u, y_v, y, but not all of them uniquely. There are in fact three possibilities for ambiguous expressions, since

$$(11)\qquad \frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \qquad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \qquad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}.$$

The second of these yields a relation of the form

$$(2)\qquad \alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0,$$

which can be satisfied only if $\alpha = \beta = \gamma = \delta = 0$. We obtain, therefore, the equations

$$
(12)\qquad
\begin{aligned}
a_v^{(21)} + b^{(21)} + a'c^{(21)} &= a_u^{(12)} + ab^{(12)} + c^{(12)},\\
a^{(21)}b^{(12)} + b_v^{(21)} + b'c^{(21)} &= a^{(12)}b^{(21)} + bb^{(12)} + b_u^{(12)} + d^{(12)},\\
a^{(21)}c^{(12)} + c'c^{(21)} + c_v^{(21)} + d^{(21)} &= a^{(12)}c^{(21)} + cb^{(12)} + c_u^{(12)},\\
a^{(21)}d^{(12)} + d'c^{(21)} + d_v^{(21)} &= a^{(12)}d^{(21)} + db^{(12)} + d_u^{(12)},
\end{aligned}
$$

* See E. J. Wilczynski, *Projective differential geometry of curved surfaces* (First Memoir), these Transactions, vol. 8 (1907), p. 239.

which must be identically satisfied. We suppose that they are; it may then be verified without difficulty that the first and third of equations (11) lead to no new relations in the presence of equations (12). All derivatives, of any order, will then be expressible, in one and only one way, as linear combinations of y_{uv}, y_u, y_v, y, in virtue of equations (12). *Equations* (12) *are a necessary and sufficient condition that system* (5) *be completely integrable, in the sense that any fundamental system of solutions of* (5) *be expressible linearly, with constant coefficients, in terms of any other fundamental system.**

It is now apparent why a completely integrable system of form (5) may be made the starting point for our projective theory. Since the configuration defined by means of any fundamental system of solutions is a projective transformation of that defined by any other fundamental system, it follows that geometric properties expressed in terms of the differential equations are common to a net and all of its projective transformations; i.e., they are projective properties.

Certain arbitrary features, however, have been introduced by our analytic formulation. Because of the use of homogeneous coördinates and parametric equations, it is possible to make any proper transformation of the dependent variable of the form

$$y = \lambda(u, v)\bar{y}, \tag{13}$$

and any proper transformation of the independent variables of the form

$$\bar{u} = U(u), \qquad \bar{v} = V(v), \tag{14}$$

without disturbing the net defined by equations (1). But these transformations change system (5) into a new system of the same form, in which the coefficients, however, are different. If, then, a combination of the coefficients and variables of system (5) is to have a geometric significance, it must remain essentially unchanged under the transformations (13) and (14). We shall call such a combination of the coefficients of (5), and their derivatives an *invariant*; if the dependent variable y also enter explicitly in the expression, we shall call it a *covariant*.

The calculation of all the invariants and covariants, as thus defined, forms the essence of Wilczynski's method. We shall not perform this calculation in the present paper, because our purpose is to seek the geometric facts, rather than to give a complete formal theory of the system of differential equations.

We deduce from the first of equations (12) a relation which will be very useful subsequently. Substituting for $b^{(21)}$, $c^{(21)}$, $b^{(12)}$, $c^{(12)}$ their values as

* In the above discussion, we have supposed that all of the functions involved are analytic. However, the statement in italics is true even if only the quantities which appear in equations (12) exist. This follows from certain theorems which we have given elsewhere. See G. M. Green, *The linear dependence of functions of several variables and certain completely integrable systems of partial differential equations*, these Transactions, vol. 17 (1916), pp. 483–516.

obtained from (9), we find that

$$a^{(21)}_v + b_v = a^{(12)}_u + c'_u. \tag{15a}$$

This shows that there exists a function f such that

$$f_u = a^{(21)} + b, \qquad f_v = a^{(12)} + c'. \tag{15b}$$

We have supposed that the parametric net is not a conjugate net, and thus were enabled to write the differential equations of the problem in the form (5). We shall find it convenient to write these equations also in the form

$$\begin{aligned} y_{uv} &= \alpha y_{uu} + \beta y_u + \gamma y_v + \delta y, \\ y_{uv} &= \alpha' y_{vv} + \beta' y_u + \gamma' y_v + \delta' y, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \alpha &= \frac{1}{a}, \quad \beta = -\frac{b}{a}, \quad \gamma = -\frac{c}{a}, \quad \delta = -\frac{d}{a}, \\ \alpha' &= \frac{1}{a'}, \quad \beta' = -\frac{b'}{a'}, \quad \gamma' = -\frac{c'}{a'}, \quad \delta' = -\frac{d'}{a'}. \end{aligned} \tag{16a}$$

Of course in doing this we assume that neither a nor a' is zero. Geometrically, this is equivalent to assuming that *neither the curves* C_u ($v = const.$) *nor the curves* C_v ($u = const.$) *are asymptotic.**

2. Certain nets determined by the given one

On the surface S_y determined by equations (1), the net of asymptotic curves is of course fundamental. This net may be found from the given net as follows. By a transformation of the independent variables of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

the given parametric net is changed into a new parametric net, which is asymptotic if and only if ϕ and ψ satisfy the same quadratic partial differential equation of the first order,†

$$a'\phi_u^2 - 2\phi_u\phi_v + a\phi_v^2 = 0, \qquad a'\psi_u^2 - 2\psi_u\psi_v + a\psi_v^2 = 0. \tag{17}$$

* The study of asymptotic nets belongs more properly to the general theory of curved surfaces, since only one asymptotic net exists on a curved surface. The theory of surfaces has in fact been developed on this basis in the first three of Wilczynski's five memoirs on curved surfaces, these Transactions, vols. 8–10 (1907–1909). Although the case in which only one of the families of curves is composed of asymptotics would appear at first sight to be less interesting on account of its lack of symmetry, it seems nevertheless to be useful in the general theory of congruences, and especially in the theory of W-congruences, the congruence consisting of the tangents to the non-asymptotic curves on the surface.

† E. J. Wilczynski, *Projective differential geometry of curved surfaces*, First Memoir, these Transactions, vol. 8 (1907), pp. 233–260, in particular p. 243.

We may express this otherwise by saying that *the asymptotic net on the surface S_y is determined by the ordinary quadratic differential equation*

$$adu^2 + 2dudv + a' dv^2 = 0, \tag{18}$$

whose factors are of course distinct, since $1 - aa' \neq 0$. At any point of the surface, equation (18) determines the two asymptotic tangents. In fact, if $(dv/du)_1$ and $(dv/du)_2$ are the two roots of the quadratic, then

$$y_u + \left(\frac{dv}{du}\right)_1 y_v, \qquad y_u + \left(\frac{dv}{du}\right)_2 y_v$$

are points on the asymptotic tangents.

In general, a quadratic differential equation

$$Adu^2 + 2Bdudv + Cdv^2 = 0, \tag{19}$$

whose discriminant is different from zero, will define a net of curves on the surface, and will determine at any point of the surface the two tangents to the curves of the net passing through that point. The coefficients are of course functions of u and v. If, then, a second net be defined by the differential equation

$$A' du^2 + 2B' dudv + C' dv^2 = 0, \tag{20}$$

the two tangents defined by (19) at any point of the surface will separate harmonically those defined by (20), if and only if the harmonic invariant of the two quadratics vanishes, i.e., if and only if

$$AC' - 2BB' + A' C = 0. \tag{21}$$

Again, the pair of tangents defined by (19) and the pair of tangents defined by (20) determine an involution, and the double rays of the involution will be defined by equating to zero the jacobian of the two quadratic forms:

$$(AB' - A' B) du^2 + (AC' - A' C) dudv + (BC' - B' C) dv^2 = 0. \tag{22}$$

The rays defined by equation (22) of course separate harmonically the pair given by (19) and the pair given by (20). We have therefore a means for determining uniquely a third net of curves from any two given ones.

Let us apply this idea to the parametric net and the asymptotic net. We thus determine a net whose tangents at any point separate harmonically the pair of parametric tangents and the pair of asymptotic tangents. But two tangents at a point which separate harmonically the asymptotic tangents at the same point are conjugate, so that the net which we have determined is a conjugate net; we shall call it the *associate conjugate net.* Its differential equation is

$$adu^2 - a' dv^2 = 0. \tag{23}$$

We may state our result in this form: *there exists one and only one conjugate net, the associate conjugate net, such that the tangents to the two curves of the net at any point separate harmonically the parametric tangents at that point.*

We have, of course, implicitly supposed that neither family of curves constituting the parametric net is composed of asymptotics, i.e., we have assumed that both a and a' are different from zero, as we did at the end of Section 1.

From the parametric net may be determined still other nets and families of curves. For instance, the family of curves C_u ($v =$ const.) determines a unique conjugate family. The conjugate net of which the curves C_u form a component family is defined by the differential equation

$$adudv + dv^2 = 0, \tag{24}$$

since $v =$ const. is a solution of this equation, and moreover the simultaneous invariant (21) of (18) and (24) is zero. Therefore, *the one-parameter family of curves conjugate to the family C_u ($v =$ const.) is determined by the differential equation*

$$adu + dv = 0. \tag{25}$$

Similarly, *the family conjugate to the family C_v ($u =$ const.) is determined by the differential equation*

$$du + a'dv = 0. \tag{26}$$

The conjugate net determined by the family of curves C_u (and also that determined by the curves C_v) will be important later. We have investigated elsewhere* the subject of conjugate nets from the point of view of the present paper, and some of the results of that investigation will be useful. Equation (25) shows that *the transformation $\bar{u} = U(u, v)$, $\bar{v} = v$, where*

$$U_u - aU_v = 0, \tag{27}$$

makes parametric the conjugate net determined by the one-parameter family of curves C_u ($v =$ const.) of the original parametric net.

We shall find it convenient to denote by $\bar{C}_v$ the curves $\bar{u} =$ const. which are conjugate to the curves C_u, and shall then speak of the conjugate net $(C_u, \bar{C}_v)$.

Let us make the transformation just indicated. We have

$$y_u = \bar{y}_u U_u, \qquad y_v = \bar{y}_u U_v + \bar{y}_v, \tag{28}$$

$$\begin{aligned} y_{uu} &= \bar{y}_{uu} U_u^2 + \bar{y}_u U_{uu}, \qquad y_{uv} = \bar{y}_{uu} U_u U_v + \bar{y}_{uv} U_u + \bar{y}_u U_{uv}, \\ y_{vv} &= \bar{y}_{uu} U_v^2 + 2\bar{y}_{uv} U_v + \bar{y}_{vv} + \bar{y}_u U_{vv}, \end{aligned} \tag{29}$$

* G. M. Green, *Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface* (First Memoir), American Journal of Mathematics, vol. 37 (1915), pp. 215–246; (Second Memoir), ibid., vol. 38 (1916), pp. 287–324.

where $\bar{y}_u$, $\bar{y}_{uu}$, etc., denote $\partial y/\partial \bar{u}$, $\partial^2 y/\partial \bar{u}^2$, etc. From the first and second of equations (29), we obtain

$$y_{uu} - a y_{uv} = - a U_u \bar{y}_{uv} + a_u U_v \bar{y}_u ,$$

in which use has been made of equation (27) and of the first of the following two equations which result from differentiation of (27):

$$(30) \qquad U_{uu} - a U_{uv} = a_u U_v , \qquad U_{uv} - a U_{vv} = a_v U_v .$$

Using now the first of equations (5), and also equations (28) and (27), and introducing the abbreviations (16a), we find very readily the equation

$$(31) \qquad \bar{y}_{uv} = \left(\frac{a_u}{a^2} + \beta + \frac{\gamma}{a} \right) \bar{y}_u + \frac{\gamma}{U_u} \bar{y}_v + \frac{\delta}{U_u} \bar{y} .$$

This is one of the two partial differential equations of the second order which constitute the completely integrable system for the surface referred to the conjugate net $(C_u, \bar{C}_v)$. The other equation may be found as follows. From the second and third of equations (29), we find

$$y_{vv} - a' y_{uv} = \bar{y}_{uu} (U_v^2 - a' U_u U_v) + \bar{y}_{uv} (2U_v - a' U_u) + \bar{y}_{vv} + \bar{y}_u (U_{vv} - a' U_{uv}) ,$$

which in virtue of (5), (27), (30), and (31) may be reduced to an equation in $\bar{y}_{uu}$, $\bar{y}_{vv}$, $\bar{y}_u$, $\bar{y}_v$, and $\bar{y}$. *If the conjugate net* $(C_u, \bar{C}_v)$ *be made parametric, the surface* S_y *is defined by a fundamental system of solutions of the completely integrable system of partial differential equations*

$$(32) \qquad \begin{aligned} \bar{y}_{uu} &= \bar{a} \bar{y}_{vv} + \bar{b} \bar{y}_u + \bar{c} \bar{y}_v + \bar{d} \bar{y} , \\ \bar{y}_{uv} &= \bar{b}' \bar{y}_u + \bar{c}' \bar{y}_v + \bar{d}' \bar{y} , \end{aligned}$$

where

$$(33) \qquad \begin{aligned} \bar{a} &= \frac{1}{(aa' - 1) U_v^2} , \\ \bar{b} &= \frac{1}{(1 - aa') U_v} \left[ab' + c' + a' a_v - (2 - aa') \left(\frac{a_u}{a^2} + \beta + \frac{\gamma}{a} \right) - (1 - aa') \frac{U_{vv}}{U_v} \right] , \\ \bar{c} &= \frac{1}{(1 - aa') U_v^2} \left(c' - \frac{2 - aa'}{a} \gamma \right) , \\ \bar{d} &= \frac{1}{(1 - aa') U_v^2} \left(d' - \frac{2 - aa'}{a} \delta \right) , \\ \bar{b}' &= \frac{a_u}{a^2} + \beta + \frac{\gamma}{a} , \qquad \bar{c}' = \frac{\gamma}{U_u} , \qquad \bar{d}' = \frac{\delta}{U_u} . \end{aligned}$$

Equations (33) are precisely of the form used in our investigations on conjugate nets, in the papers already cited. In order to make use of the results of these papers, however, it will be necessary to note certain differentiation formulas. From (28) we have, on using (27),

$$\frac{\partial}{\partial \bar{u}} = \frac{1}{U_u}\frac{\partial}{\partial u}, \qquad \frac{\partial}{\partial \bar{v}} = \frac{\partial}{\partial v} - \frac{1}{a}\frac{\partial}{\partial u}, \tag{34}$$

whence

$$\frac{\partial^2}{\partial \bar{u}\partial \bar{v}} = \frac{1}{U_u}\left(\frac{\partial^2}{\partial u \partial v} - \frac{1}{a}\frac{\partial^2}{\partial u^2} + \frac{a_u}{a^2}\frac{\partial}{\partial u}\right),$$

$$\frac{\partial^2}{\partial \bar{v}^2} = \frac{\partial^2}{\partial v^2} - \frac{2}{a}\frac{\partial^2}{\partial u \partial v} + \frac{1}{a^2}\frac{\partial^2}{\partial u^2} + \left(\frac{a_v}{a^2} - \frac{a_u}{a^3}\right)\frac{\partial}{\partial u}. \tag{35}$$

The formulas

$$\frac{\partial U_u}{\partial \bar{v}} = -\frac{a_u}{a}U_v, \qquad \frac{\partial U_v}{\partial \bar{v}} = -\frac{a_v}{a}U_v, \tag{35a}$$

which follow from (34) and (30), will also be found convenient later.

We may now express, in terms of the coefficients and variables of equations (5), some of the important invariants of the conjugate net $(C_u, \bar{C}_v)$. These invariants are given explicitly in terms of the differential equations (32), in our memoirs on conjugate nets. The Laplace-Darboux invariants of the net are*

$$H = \bar{d}' + \bar{b}'\bar{c}' - \bar{b}'_u, \qquad K = \bar{d}' + \bar{b}'\bar{c}' - \bar{c}'_v, \tag{36}$$

so that, writing

$$H = -\frac{1}{aU_u}h, \qquad K = -\frac{1}{aU_u}k, \tag{37}$$

we find in virtue of (33) and (34) that

$$h = d - \gamma\left(2\frac{a_u}{a} + a\beta + \gamma\right) + \gamma_u - a(\alpha_{uu} - \beta_u),$$

$$k = d - \gamma(\gamma + a\beta) - \gamma_u + a\gamma_v. \tag{38}$$

In the first memoir on conjugate nets we denoted by $\mathfrak{B}'$ and $\mathfrak{C}'$ two invariants which, if use be made of equations (24), (29), and (38) of that memoir, are without difficulty seen to have the values

$$\bar{\mathfrak{B}}' = \frac{1}{4}\left(2\bar{b}' + \frac{\bar{c}}{\bar{a}} - \frac{\bar{a}_v}{2\bar{a}}\right), \qquad \bar{\mathfrak{C}}' = \frac{1}{4}\left(2\bar{c}' - \bar{b} + \frac{\bar{a}_u}{2\bar{a}}\right). \tag{39}$$

We have of course written bars over the $\mathfrak{B}'$ and $\mathfrak{C}'$ in accordance with our present notation. We may express the right-hand members of equations (39) in terms of the coefficients and variables of the original system of differential equations (5), by using formulas (33) and (34); writing

* First Memoir, equations (42), (47).

(40)
$$\bar{\mathfrak{C}}' = \frac{1}{U_v(1 - aa')}\{\bar{\mathfrak{C}}'\},$$

we find that

(41)
$$4\bar{\mathfrak{B}}' = 2\beta + \frac{4 - aa'}{a}\gamma - c' + 2\frac{a_u}{a^2} - \frac{a_v}{a} + \frac{1}{2}\frac{\partial}{\partial v}\log(1 - aa') - \frac{1}{2a}\frac{\partial}{\partial u}\log(1 - aa'),$$
$$4\{\bar{\mathfrak{C}}'\} = (2 - aa')\beta + \frac{4 - 3aa'}{a}\gamma - ab' - c' + (2 - aa')\frac{a_u}{a^2} - \frac{a_v}{a} - \frac{1 - aa'}{2a}\frac{\partial}{\partial u}\log(1 - aa').$$

The invariants $\bar{\mathfrak{B}}'$ and $\bar{\mathfrak{C}}'$ are, strictly speaking, invariants peculiar to the conjugate net $(C_u, \bar{C}_v)$; for instance, the equation $\bar{\mathfrak{B}}' = 0$ expresses a geometric property of this net* which is independent of the nature of the surface, in the sense that on any surface whatever there exist conjugate nets for which $\bar{\mathfrak{B}}' = 0$. Certain combinations of the invariants $\bar{\mathfrak{B}}'$ and $\bar{\mathfrak{C}}'$, however, express intrinsic properties of the surface itself. For instance, if and only if one of the quantities $\bar{\mathfrak{C}}' + \sqrt{-\bar{a}\bar{\mathfrak{B}}'}$, $\bar{\mathfrak{C}}' - \sqrt{-\bar{a}\bar{\mathfrak{B}}'}$ vanishes, i.e., if and only if $\bar{\mathfrak{C}}'^2 + \bar{a}\bar{\mathfrak{B}}'^2 = 0$, will the surface be a ruled surface; also, the surface will be a quadric if and only if both $\bar{\mathfrak{B}}'$ and $\bar{\mathfrak{C}}'$ are zero.†

In precisely the same way, we may set up the corresponding invariants for the conjugate net $(\bar{C}_u, C_v)$ formed by the family of curves C_v (u = const.) and the family conjugate thereto. The formulas may be written down, however, by changing each of the invariants already found for the conjugate net $(C_u, \bar{C}_v)$ in accordance with the transformation scheme

$$\left\{\begin{matrix} u, & a, & b, & c, & d, & \alpha, & \beta, & \gamma, & \delta \\ v, & a', & c', & b', & d', & \alpha', & \gamma', & \beta', & \delta' \end{matrix}\right\},$$

any letter being replaced by the letter immediately above or below it. Thus, for instance, the Laplace-Darboux invariants—or rather the invariants corresponding to the quantities (38)—are for the conjugate net $(\bar{C}_u, C_v)$

(42)
$$h' = d' - \beta'(\beta' + a'\gamma') - \beta'_v + a'\beta'_u,$$
$$k' = d' - \beta'\left(2\frac{a'_v}{a'} + a'\gamma' + \beta'\right) + \beta'_v - a'(\alpha'_{vv} - \gamma'_v).$$

---

* See page 317 of the second memoir on conjugate nets for the geometric interpretation of the vanishing of either $\bar{\mathfrak{B}}'$ or $\bar{\mathfrak{C}}'$.

† See the first memoir on conjugate nets, p. 239.

3. The congruences of tangents to the families of the net. Laplace transformations

We recall that on the surface S_y, given by the equations

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1 bis}$$

the parametric curves are not asymptotic, and do not form a conjugate net. Let us consider the congruence of tangents to the curves C_u (v = const.). One sheet of its focal surface is the surface S_y; the other sheet may be determined as follows. The point whose coördinates are $y_u^{(1)}, \cdots, y_u^{(4)}$, or, as we may say, the point y_u, is on the tangent to the curve C_u which passes through the point y. Any point on this tangent except the point y is given by the formula

$$\rho = y_u + \lambda y. \tag{43}$$

Consider now the ruled surface whose generators are the tangents to the curves C_u at the points where these curves meet a fixed curve C_v. This ruled surface is not developable, since we have supposed that the curves C_u and C_v are not conjugate. Now, any plane passing through a non-specialized generator of a non-developable ruled surface is tangent to the surface at some point on that generator. The point ρ describes of course a curve on the ruled surface; we shall now determine λ as a function of u and v in such a way that the curve traced by ρ shall be the locus of the points in which the ruled surface is touched by the planes which osculate the curves C_u at the points of the fixed curve C_v. We need only express the condition that the tangent to the curve ρ at any point lies in the plane which osculates at the corresponding point y the corresponding curve C_u. The point ρ is on this tangent; another point on the tangent to the curve traced by ρ as v varies along the fixed curve C_v is given by

$$\rho_v = y_{uv} + \lambda y_v + \lambda_v y,$$

which by the first of equations (16) becomes

$$\rho_v = \alpha y_{uu} + \beta y_u + (\lambda + \gamma) y_v + (\lambda_v + \delta) y.$$

Now, the points y, y_u, y_{uu} determine the osculating plane to the curve C_u at y, therefore the point ρ_v lies in this osculating plane if and only if $\lambda = -\gamma$. This gives us the required expression for ρ, viz.

$$\rho = y_u - \gamma y. \tag{44}$$

We may describe the point ρ geometrically in a slightly different way. Consider two points y and $y + \Delta y$ on a single curve C_v, and the two curves C_u and C'_u which pass through them. The tangent to C_u at y cuts in a point P the plane which osculates C'_u at $y + \Delta y$. As $y + \Delta y$ approaches y along C_v, the point P approaches the point ρ. If we note that the osculating planes to C_u and C'_u at y and $y + \Delta y$ intersect in a line which passes through P, we may give the following geometric characterization of the quantity ρ. *The tangents to the curves C_u at the points where they meet a fixed curve C_v form a non-developable ruled surface. The osculating planes to the curves C_u at the same points envelop a developable surface. The ruled surface is tangent to the developable surface along a curve which is given by the equations*

$$\rho^{(k)} = y_u^{(k)} - \gamma y^{(k)} \qquad (k = 1, 2, 3, 4)$$

for u = const.

For each point y on the surface S_y, we have a corresponding point ρ, and consequently the totality of points ρ form in general a surface S_ρ. We shall consider the exceptional cases presently. That the surface S_ρ is the second sheet of the focal surface of the congruence of tangents to the curves C_u (v = const.) on S_y follows immediately from the above geometric discussion. In fact, as is well known, the osculating planes to the curves C_u envelop the second focal sheet of the congruence of tangents to these curves. We may also prove this directly. The points ρ, ρ_u, ρ_v determine the tangent plane to the surface S_ρ at ρ. We have

$$\begin{aligned} \rho_u &= y_{uu} - \gamma y_u - \gamma_u y, \\ \rho_v &= \alpha y_{uu} + \beta y_u + (\delta - \gamma_v) y, \end{aligned} \tag{45}$$

so that the said tangent plane is in fact the osculating plane, at the corresponding point y, to the curve v = const. The line joining y and ρ lies in this plane, so that this line is tangent to the surface S_ρ at ρ. *The surface S_ρ is the second focal sheet of the congruence of tangents to the curves C_u (v = const.) on the surface S_y.*

Similarly, *the second focal sheet of the congruence of tangents to the curves C_v (u = const.) on S_y is the surface S_σ defined by the quantity*

$$\sigma = y_v - \beta' y. \tag{46}$$

The analogy of the foregoing to the theory of the Laplace transformations of surfaces referred to conjugate nets, as developed by Darboux, is obvious. To the original surface S_y we have made to correspond, by means of equation (44), the surface S_ρ, and to the net of parameter curves on S_y a net u = const., v = const. on S_ρ. We may regard equation (44) as defining a transformation of S_y into S_ρ. In the same way, equation (46) transforms S_y into S_σ. Now, the same operations may be carried out on the surfaces S_ρ and S_σ, giving rise to two new surfaces for each of these. And so on indefinitely. There is, however, no single continuous sequence as in the Darboux theory, because our original net of curves is not conjugate. Nor can a parametric net on any one of the surfaces derived from S_y by repeated application of the transformations (44) and (46) ever be conjugate, if the surface is non-degenerate and non-developable, because in that case, by Darboux's theory, the parametric net on the surface from which the conjugate net was derived would also be conjugate. It follows that if S_η be any one of the set of non-degenerate, non-developable surfaces derived from S_y by repeated application of the transformations (44) and (46), then the points $\eta_{uv}, \eta_u, \eta_v, \eta$ must be linearly independent, that is, there must exist between these quantities no relation of the form (2), with non-vanishing coefficients:

$$A\eta_{uv} + B\eta_u + C\eta_v + D\eta = 0.$$

It may happen, however, that such a relation occurs for a surface of the series. That surface must then be developable or degenerate. We investigate this possibility for the surface S_ρ, by obtaining the expressions for ρ, ρ_u, ρ_v, and ρ_{uv}:

$$(47)\quad \begin{aligned} \rho &= y_u - \gamma y, \qquad \rho_u = a y_{uv} + (b-\gamma) y_u + c y_v + (d - \gamma_u) y, \\ \rho_v &= y_{uv} - \gamma y_v - \gamma_v y, \\ \rho_{uv} &= y_{uuv} - \gamma y_{uv} - \gamma_u y_v - \gamma_v y_u - \gamma_{uv} y \\ &= (a^{(21)} - \gamma) y_{uv} + (b^{(21)} - \gamma_v) y_u \\ &\qquad + (c^{(21)} - \gamma_u) y_v + (d^{(21)} - \gamma_{uv}) y, \end{aligned}$$

where $a^{(21)}$, $b^{(21)}$, $c^{(21)}$, $d^{(21)}$ are given by equations (9). We have supposed y, y_u, y_v, y_{uv} linearly independent, so that the determinant of their coefficients in equations (47) must vanish if a relation of form (2) is to exist between $\rho, \rho_u, \rho_v, \rho_{uv}$; we have, then,

$$\begin{vmatrix} 0 & 1 & 0 & -\gamma \\ a & b-\gamma & c & d-\gamma_u \\ 1 & 0 & -\gamma & -\gamma_v \\ a^{(21)}-\gamma & b^{(21)}-\gamma_v & c^{(21)}-\gamma_u & d^{(21)}-\gamma_{uv} \end{vmatrix} = 0.$$

This condition is very easily found to reduce to

$$(48)\quad [d + \gamma(b-\gamma) - \gamma_u + a\gamma_v](c^{(21)} + \gamma a^{(21)} - \gamma_u - \gamma^2) = 0.$$

Each factor of the left-hand member is of course an invariant; in fact, the first one is the Laplace-Darboux invariant k of the conjugate net $(C_u, \overline{C}_v)$, given by the second of equations (38). We may state then the result, that *the surface S_ρ is developable or degenerate if either of the invariants*

$$(49)\quad \begin{aligned} k &= d + \gamma(b-\gamma) - \gamma_u + a\gamma_v, \\ G &= c^{(21)} + \gamma a^{(21)} - \gamma_u - \gamma^2 \end{aligned}$$

vanish for the original net on S_y.

Similarly, *the surface S_σ is developable or degenerate if either of the invariants*

$$
(50)\qquad \begin{aligned} h' &= d' + \beta'(c' - \beta') - \beta'_v + a'\beta'_u, \\ G' &= b^{(12)} + \beta' a^{(12)} - \beta'_v - \beta'^2 \end{aligned}
$$

vanish.

The geometric interpretation of the vanishing of G or G' is not difficult. We have by (8)

$$y_{uuu} = a^{(30)} y_{uv} + b^{(30)} y_u + c^{(30)} y_v + d^{(30)} y,$$

which on using the first of equations (16) becomes

$$y_{uuu} = \alpha a^{(30)} y_{uu} + (b^{(30)} + \beta a^{(30)}) y_u + (c^{(30)} + \gamma a^{(30)}) y_v + (d^{(30)} + \delta a^{(30)}) y.$$

*The curves C_u are plane if and only if the quantity $c^{(30)} + \gamma a^{(30)}$ vanishes.** But by using the values of $c^{(30)}$ and $a^{(30)}$ as given by equations (9), we find that

$$(51)\qquad c^{(30)} + \gamma a^{(30)} = a(c^{(21)} + \gamma a^{(21)} - \gamma_u - \gamma^2) = aG.$$

The vanishing of the invariant G is a necessary and sufficient condition that the curves v = const. on S_y be plane. The curves u = const. on S_y are plane if and only if the invariant G' vanishes.

If the curves C_u are plane curves, the surface S_ρ is the developable surface enveloped by the planes of these curves; if the curves C_v are plane, the surface S_σ is the developable enveloped by their planes.

The interpretation of the vanishing of the Laplace-Darboux invariant k is well-known. The surface S_ρ in this case degenerates into a curve, and the curves C_u are then conjugate to a family of curves of contact of cones enveloping the surface and having their vertices on a curve in space, viz. the curve into which S_ρ degenerates.†

The cases just considered are the only ones in which the sequence of Laplace transformations for a surface referred to a conjugate net is stopped in either direction. For our surface S_y, however, which is referred to a non-conjugate net, the series of surfaces obtained by successive application of the transformations (44) and (46) may be stopped in other ways. First we note that we have already disposed of the cases in which the quantities ρ_{uv}, ρ_u, ρ_v, ρ are connected by a linear relation. We may therefore suppose that the determinant (3) formed for ρ does not vanish in the case we shall now consider. Then the quantity ρ will satisfy a completely integrable system of form (5). If we eliminate between equations (47) and the equation

* For, if $c^{(30)} + \gamma a^{(30)} = 0$, a linear relation exists between y_{uuu}, y_{uu}, y_u, y; i.e., the determinant $|y^{(k)}, y_u^{(k)}, y_{uu}^{(k)}, y_{uuu}^{(k)}|$ vanishes identically. But for v = const., this determinant is nothing but the wronskian of the four functions $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$. Therefore for a fixed v a linear relation with *constant* coefficients exists between $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$.

† Cf. the first memoir on conjugate nets, p. 236.

$$
\begin{aligned}
\rho_{uu} &= y_{uuu} - \gamma y_{uu} - 2\gamma_u y_u - \gamma_{uu} y \\
&= (a^{(30)} - a\gamma) y_{uv} + (b^{(30)} - b\gamma - 2\gamma_u) y_u + (c^{(30)} - c\gamma) y_v \\
&\qquad + (d^{(30)} - d\gamma - \gamma_{uu}) y
\end{aligned} \tag{52}
$$

the quantities y_{uv}, y_u, y_v, y—an elimination which may be made in only one way, since no linear relation connects these quantities—we obtain the first of the following equations:

$$
\begin{aligned}
\rho_{uu} &= a_{-1}\,\rho_{uv} + b_{-1}\,\rho_u + c_{-1}\,\rho_v + d_{-1}\,\rho, \\
\rho_{vv} &= a'_{-1}\,\rho_{uv} + b'_{-1}\,\rho_u + c'_{-1}\,\rho_v + d'_{-1}\,\rho,
\end{aligned} \tag{53}
$$

where in particular

$$
a_{-1} = a. \tag{54}
$$

The second of equations (53) is obtained by eliminating y_{uv}, y_u, y_v, y between equations (47) and the equation

$$
\begin{aligned}
\rho_{vv} = (a^{(12)} - a'\gamma) y_{uv} &+ (b^{(12)} - b'\gamma) y_u \\
&+ (c^{(12)} - c'\gamma - 2\gamma_v) y_v + (d^{(12)} - d'\gamma - \gamma_{vv}) y,
\end{aligned} \tag{55}
$$

and the coefficient a'_{-1} is without much difficulty found to have the value

$$
a'_{-1} = \frac{F}{G}, \tag{56}
$$

where G is given by (49), and

$$
F = c^{(12)} + \gamma a^{(12)} - c'\gamma - a'\gamma^2 - 2\gamma_v. \tag{57}
$$

In order, now, that the surface S_ρ be non-developable, we must have $1 - a_{-1} a'_{-1}$ different from zero, i.e., $G - aF$ must not vanish. On using the values for $a^{(12)}$, $c^{(12)}$, $a^{(21)}$, $c^{(21)}$ given by (9), we find that

$$
G - aF = k, \tag{58}
$$

so that the vanishing of this quantity would lead to nothing new.

There remains, however, the possibility that a'_{-1}, in other words F, may vanish. In this case the curves $u =$ const. on the surface S_ρ are asymptotics, and it is obvious that the congruence of tangents to these curves can have only the one focal sheet S_ρ. We may therefore state the result, that *aside from the cases already considered, in which a surface of the series derived from S_y by means of the transformations* (44) *and* (46) *is developable or degenerate, the only case in which the series is stopped in any direction arises when a parametric family of curves on a surface derived from S_y is asymptotic. The curves $v =$ const. on S_ρ, and $u =$ const. on S_σ, cannot be asymptotic; the curves $u =$ const. on S_ρ are asymptotic if and only if the invariant F given by* (57) *is zero, while neither k nor G vanishes. Similarly, if and only if the invariant*

$$F' = b^{(21)} + \beta' a^{(21)} - b\beta' - a\beta'^2 - 2\beta'_u \tag{59}$$

vanishes, whereas h' and G' are neither of them zero, will the curves v = const. on S_σ be asymptotic.

In terms of invariants just obtained may be expressed two very important ones. A W-congruence is a congruence for which the asymptotic curves on the two focal sheets correspond. We may therefore obtain the condition that the tangents to the curves C_u on S_y constitute a W-congruence by writing down the condition that the differential equation which determines the asymptotic net on S_y is the same as the differential equation which determines the asymptotic net on S_ρ. The asymptotics on S_y are determined by equation (18); the asymptotics on S_ρ are therefore determined by the equation

$$a_{-1}\, du^2 + 2du dv + a'_{-1}\, dv^2 = 0. \tag{60}$$

Recalling that $a_{-1} = a$, we see at once that equations (18) and (60) are identical if and only if $a'_{-1} = a'$. Using equation (56), we find that this is equivalent to the vanishing of the invariant

$$W^{(u)} = F - a' G; \tag{61}$$

and by means of the expressions given by equations (9) for $a^{(12)}$, $c^{(12)}$, $a^{(21)}$, $c^{(21)}$, we find without much difficulty that

$$W^{(u)} = \frac{\partial}{\partial u}(a' \gamma + c') - 2\gamma_v. \tag{62}$$

A necessary and sufficient condition that the congruence of tangents to the curves C_u (v = const.) on the surface S_y be a W-congruence, is the vanishing of the invariant $W^{(u)}$ defined by equation (61) *or* (62).

Similarly, *the tangents to the curves C_v (u = const.) on S_y form a W-congruence if and only if the invariant*

$$W^{(v)} = F' - aG' = \frac{\partial}{\partial v}(a\beta' + b) - 2\beta'_u \tag{63}$$

vanishes.

4. The axis congruence

In analogy to the moving trihedral of metric differential geometry, it is important to have for our configuration a moving tetrahedron of reference. We have already found what may serve as three vertices of the tetrahedron, viz. the points y, ρ, σ, and the two edges $y\rho$ and $y\sigma$. We now determine the third edge through y, and on it the fourth vertex of the tetrahedron, as follows.

Since the two curves C_u and C_v passing through a point y are not asymptotic, their osculating planes at the point y intersect in a line passing through y

and not lying in the tangent plane to the surface at y. This line we shall take as a third edge of our tetrahedron; we shall call it the *axis** of the point y. It remains, therefore, to find thereon a covariant point to serve as the fourth vertex.

We first define the axis analytically. If we equate the two values of y_{uv} given by equations (16), and then make suitable transpositions, we find that

$$\alpha y_{uu} + (\beta - \beta')y_u + \delta y = \alpha' y_{vv} + (\gamma' - \gamma)y_v + \delta' y.$$

The left-hand member represents a point in the osculating plane to the curve C_u at y, the right-hand member a point in the osculating plane to the curve C_v at y. Consequently either of these expressions represents some point z on the line of intersection of the two osculating planes. Using equations (5) and (16*a*), we may find an unambiguous expression for this point: *the axis of the point y on the surface S_y is the line joining the point y with the point z determined by the expression*

$$z = y_{uv} - \beta' y_u - \gamma y_v. \tag{64}$$

To each point y of the surface corresponds a definite axis yz. The totality of axes form a congruence, which we shall call the *axis congruence.* We may then determine the two sheets of the focal surface of the axis congruence, and thereby obtain on any line yz of the congruence two covariant points, viz., the focal points of the line yz. We shall find presently that the analytic expressions for these focal points are rather complicated, and in fact involve irrationalities; we obtain, however, a unique covariant point by taking the harmonic conjugate of y with respect to the two focal points. That this choice is a very natural one, and one which leads to some important consequences, will appear in the sequel.

Any point on the line yz, with the exception of the point y, is given by the expression

$$\eta = z + \lambda y = y_{uv} - \beta' y_u - \gamma y_v + \lambda y.$$

We wish first to determine λ as a function of u and v in such a way that the surface S_η thus defined shall be a focal sheet of the axis congruence. There will therefore in general be two such functions $\lambda(u, v)$. The tangent plane to the surface S_η at any point η is determined by the three points η, η_u, η_v, so that if S_η is to be a focal sheet of the axis congruence this plane must contain in it the line yz, or in other words the point y. Therefore we must obtain the condition that the four quantities y, η, η_u, η_v be linearly dependent, or what is the same thing, that y, z, η_u, η_v be linearly dependent. We have

* This name was introduced by Wilczynski for the corresponding case in which the parameter net on S_y is conjugate. See his paper *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 314.

$$
\begin{aligned}
z_u &= y_{uuv} - \beta' y_{uu} - \gamma y_{uv} - \beta'_u y_u - \gamma_u y_v \\
&= (a^{(21)} - a\beta' - \gamma) y_{uv} + (b^{(21)} - b\beta' - \beta'_u) y_u \\
&\qquad + (c^{(21)} - c\beta' - \gamma_u) y_v + (d^{(21)} - d\beta') y, \\
z_v &= (a^{(12)} - a'\gamma - \beta') y_{uv} + (b^{(12)} - b'\gamma - \beta'_v) y_u \\
&\qquad + (c^{(12)} - c'\gamma - \gamma_v) y_v + (d^{(12)} - d'\gamma) y,
\end{aligned}
\tag{65}
$$

so that

$$
\begin{aligned}
\eta_u &= (a^{(21)} - a\beta' - \gamma) y_{uv} + (b^{(21)} - b\beta' - \beta'_u + \lambda) y_u \\
&\qquad + (c^{(21)} - c\beta' - \gamma_u) y_v + (d^{(21)} - d\beta' + \lambda_u) y, \\
\eta_v &= (a^{(12)} - a'\gamma - \beta') y_{uv} + (b^{(12)} - b'\gamma - \beta'_v) y_u \\
&\qquad + (c^{(12)} - c'\gamma - \gamma_v + \lambda) y_v + (d^{(12)} - d'\gamma + \lambda_v) y.
\end{aligned}
\tag{65a}
$$

Also,

$$z = y_{uv} - \beta' y_u - \gamma y_v .$$

Now, the points y, z, η_u, η_v will lie in a plane if and only if the determinant of the coefficients of y_{uv}, y_u, and y_v in the above expressions for η_u, η_v, and z is zero. This determinant when expanded leads to a quadratic equation in λ:

$$
\begin{aligned}
&\lambda^2 + (F + F' + \beta'_u + \gamma_v - 2\beta'\gamma)\lambda \\
&\qquad + (F + \gamma_v - \beta'\gamma)(F' + \beta'_u - \beta'\gamma) - GG' = 0,
\end{aligned}
\tag{66}
$$

where F, F', G, and G' are given by (49), (50), (57), and (59). *The focal sheets of the axis congruence are given by the expressions*

$$\tau_1 = z + \lambda_1 y, \qquad \tau_2 = z + \lambda_2 y, \tag{67}$$

where λ_1 *and* λ_2 *are the roots of the quadratic* (66).

The actual explicit expressions for λ_1 and λ_2 are of course complicated. We may, however, obtain a covariant point with a neater expression, and which is moreover uniquely determined, by choosing the point τ which is the harmonic conjugate of the point y with respect to the corresponding focal points τ_1, τ_2. We find without difficulty that

$$
\begin{aligned}
\tau &= z + \tfrac{1}{2}(\lambda_1 + \lambda_2) y \\
&= z - \tfrac{1}{2}(F + F' - 2\beta'\gamma + \beta'_u + \gamma_v) y.
\end{aligned}
\tag{68}
$$

The discriminant of the quadratic (66) is easily found to be

$$\Delta = (F - F' + \gamma_v - \beta'_u)^2 + 4GG'. \tag{69}$$

Its vanishing is of course a necessary and sufficient condition that the focal sheets of the axis congruence coincide, in other words that the axis congruence consists of the tangents to one of the two families of asymptotics on the focal surface. It is obvious that on any curved surface there exist nets for which the axis congruence has this property; it will also follow from our subsequent discussion that a given congruence formed of the tangents to a family of asymptotics on a surface is the axis congruence of an indefinite number of nets of curves on any surface cutting the lines of the congruence.

5. The ruled surfaces of the axis congruence

The lines yz of the axis congruence which pass through the points of a fixed curve C_u (v = const.) on the surface S_y generate a ruled surface $R^{(u)}$. Similarly, to every curve C_v (u = const.) on S_y corresponds a ruled surface $R^{(v)}$ of the axis congruence. We shall call the surfaces $R^{(u)}$ and $R^{(v)}$ the *parametric ruled surfaces* of the axis congruence.

The osculating plane to a curve C_u at any point y contains in it the axis of that point, so that the said osculating plane is tangent to the ruled surface $R^{(u)}$ at y. Therefore, *the surface S_y intersects the parametric ruled surfaces in asymptotic curves on the latter.*

If a parametric ruled surface $R^{(u)}$ be a developable, then the osculating plane to the curve C_u at any point is tangent to $R^{(u)}$ along the entire generator through that point, so that the developable is the envelope of the osculating planes of C_u. But the axes yz corresponding to the points of C_u can not be tangents of the curve C_u, as they would have to be if C_u were a twisted curve. Consequently, *the ruled surfaces $R^{(u)}$ will be developable if and only if the corresponding curve C_u is a plane curve, and the plane of the curve will be the developable.*

Let us suppose that the ruled surfaces $R^{(u)}$ are not developable, and let us fix our attention on a single point y of the surface S_y. Through this point passes a single curve C_v, and also a single parametric ruled surface $R^{(u)}$. Then the plane which osculates the curve C_v at y passes through the generator yz and is therefore tangent to the ruled surface $R^{(u)}$ at some point of yz. We proceed to find this point of tangency. Any point of the line yz, except the point y, is given by an expression of the form

$$\eta = z + \lambda y .$$

As the point y moves along the curve C_u, the line generates the ruled surface $R^{(u)}$, and the point η traces a curve on this ruled surface. We wish to determine λ in such a way that the tangent to the curve described by η shall lie in the plane which osculates the curve C_v at y. Using the first of equations (65a), and the second of equations (16), we find without difficulty that

$$
(70)\quad \begin{aligned}
\eta_u = \alpha'(a^{(21)} - a\beta' - \gamma)y_{vv} \\
&+ [b^{(21)} - b\beta' - \beta'_u + \beta'(a^{(21)} - a\beta' - \gamma) + \lambda]y_u \\
&+ [c^{(21)} - c\beta' - \gamma_u + \gamma'(a^{(21)} - a\beta' - \gamma)]y_v \\
&+ [d^{(21)} - d\beta' + \delta'(a^{(21)} - a\beta' - \gamma) + \lambda_u]y.
\end{aligned}
$$

Now η_u is a point on the tangent to the curve traced out by η, while the osculating plane to the curve C_v at y is determined by the points y, y_v, y_{vv}. Therefore the point η_u lies in this plane if and only if the coefficient of y_u in the expression just found for η_u is zero. This condition determines λ, which according to (59) has then the value $-F' + \beta'\gamma - \beta'_u$. *The osculating plane to a curve C_v (u = const.) at a point y of the surface S_y is tangent at the point*

$$(71)\qquad \eta = z - (F' - \beta'\gamma + \beta'_u)y$$

to the parametric ruled surface $R^{(u)}$ which passes through the point y.

Similarly, *the osculating plane to the curve C_u at the point y is tangent to the ruled surface $R^{(v)}$ at the point*

$$(72)\qquad \eta' = z - (F - \beta'\gamma + \gamma_v)y.$$

The harmonic conjugate of the point y with respect to the two covariant points η and η' is precisely the point τ given by equation (68). We recall that τ was first defined as the harmonic conjugate of the point y with respect to the focal points on the line yz of the axis congruence. It is therefore natural to inquire under what conditions the points η, η' coincide with the focal points. Of course if one of them is a focal point the other must be also.

Consider the surface S_η formed by all the points η corresponding to the points of the surface S_y. The surface S_η is a focal sheet of the axis congruence if and only if the tangent plane at a point η contains in it the line yz passing through that point. In other words, we seek the condition that the points y, z, η_u, η_v lie in a plane. If in equations (65*a*) we put for λ its value, $-(F' - \beta'\gamma + \beta'_u)$, and recall that

$$z = y_{uv} - \beta' y_u - \gamma y_v,$$

we obtain the following expressions for η_u and η_v:

$$
(73)\quad \begin{aligned}
\eta_u &= (a^{(21)} - a\beta' - \gamma)z + Gy_v + (\quad)y, \\
\eta_v &= (a^{(12)} - a'\gamma - \beta')z + G'y_u + (\quad)y_v + (\quad)y,
\end{aligned}
$$

where the coefficient of y in the first equation, and the coefficients of y_v and y in the second equation, do not concern us. The quantities G and G' are the invariants given by (49) and (50), whose vanishing is a necessary and sufficient condition that the curves C_u and C_v be respectively plane curves.

Equations (73) show that the points y, z, η_u, η_v lie in the same plane if and only if $GG' = 0$. We have therefore the result: *the surfaces S_η and $S_{\eta'}$ are the focal sheets of the axis congruence if and only if one of the families of curves of the parametric net C_u, C_v on the surface S_y consist of plane curves.*

We see, then, that the points η, η' do not in general coincide with the focal points, and in recapitulation we may describe our configuration as follows. Suppose that neither of the component families C_u, C_v of the given net of curves on the surface S_y consists of plane curves or asymptotics.* At any point y of the surface, the osculating planes to the curves C_u, C_v which pass through that point intersect in a line, the axis of the point y, which is not tangent to the surface at y. The totality of these lines form a congruence, the axis congruence, the two focal sheets of which are touched by each line of the congruence in two focal points on that line. The harmonic conjugate of the point y with respect to the two focal points gives a point τ covariantly associated with y. Consider now the curves C_u and C_v which pass through the point y. The lines of the axis congruence which cut the curve C_u form a non-developable ruled surface $R^{(u)}$, and those which cut the curve C_v form a non-developable ruled surface $R^{(v)}$. The osculating plane to the curve C_v at y is tangent to the ruled surface $R^{(u)}$ at a point η on the axis of y, and the osculating plane to the curve C_u at y is tangent to the ruled surface $R^{(v)}$ at a point η' on the same axis. The harmonic conjugate of the point y with respect to the points η, η' is the point τ, which is also the harmonic conjugate of y with respect to the focal points of the axis. That is, the points η, η' are harmonically separated by the points y, τ, and these in turn by the focal points.

The above discussion suggests at once some new investigations in the general theory of congruences. First of all, given a rectilinear congruence Γ, and a surface S_y such that one and only one line yz of the congruence passes through each point y of the surface, can a net of curves be determined on S_y for which the congruence Γ is the axis congruence? The question is identical with a familiar one in metric geometry, viz., the determination of the geodesics on a surface. Recalling that a geodesic is characterized by the property that the osculating plane to the curve at any point contains in it the normal to the surface at that point, we see that if the given congruence Γ is the congruence of normals to the surface, this congruence is the axis congruence for any net of curves on the surface formed of geodesics. Our general

* In our analytic work, we have found it convenient to suppose that the net is not conjugate. Our geometric description will, however, be applicable to a conjugate net, as we have in fact shown in the second memoir on conjugate nets already cited.

problem, then, is the determination of all the curves on the surface S_y which have the property that the osculating plane at any point of one of the curves contains in it that line yz of the congruence Γ which passes through the point. It may be shown, exactly as in the case of geodesics, that there exist on the surface S_y a two-parameter family of curves, or as we may say ∞^2 curves, which have the desired property.* Any net formed of two one-parameter families of such curves will then have the congruence Γ as its axis congruence.

Let us select a net of curves of the kind just described. We may apply the geometric results obtained above to this net and its axis congruence, Γ, and thus fix, on the line of the congruence passing through y, the two points η, η'. By choosing a different net from among our ∞^2 curves on S_y, or by considering an entirely different surface through y—referred of course to a net of curves bearing the above characteristic relation to the congruence Γ—we obtain a new pair of points η, η' on the line through y. In this way, keeping y fixed, we obtain on the line an infinite number of pairs of points η, η', which, as we know, are pairs of an involution. The double points of the involution are the point y and the associated point τ. Similarly, for any other point y of the same line of the congruence may be obtained a related point τ. The associated points y and τ are in fact pairs of an involution, the double points of which are the focal points of the congruence.

Of course, the above description will lend itself to refinement, but since our discussion in the present paper is centered about a net of curves, we shall not examine more closely these questions in the general theory of congruences.

The considerations of the present section are readily exemplified in metric geometry. As we have already pointed out, if the parametric net of curves on the surface is composed of geodesics, the axis congruence is the congruence of normals to the surface. The focal sheets of this congruence are the surfaces of centers, or, in other words, the focal points are the principal centers of curvature. It is hardly necessary to state the theorems which result here from our general discussion. One of them, however, deserves explicit mention. If R_1 and R_2 denote the principal radii of curvature at a point y of the

* We have suggested here a new class of curves on a surface, which form a very natural projective generalization of geodesics. Given a surface, we may replace the congruence of its normals by another congruence also uniquely determined by the surface, but in a purely projective way. We refer to the directrix congruence of the second kind—a congruence introduced by Wilczynski in his second memoir on curved surfaces, these Transactions, vol. 9 (1908), pp. 79–120. The ∞^2 geodesics will then be replaced by the ∞^2 curves which are characterized by the property that the osculating plane to any one of the curves at any point contains the line of the directrix congruence of the second kind passing through that point.

The present writer has learned that Miss Pauline Sperry has investigated independently this very generalization of geodesics, in her Chicago dissertation, published in the American Journal of Mathematics, vol. 40 (1918), pp. 213–224.

surface, then the focal points τ_1 and τ_2 of the normal to the surface at y are at distances R_1 and R_2 from y. The point τ, which is the harmonic conjugate of y with respect to τ_1 and τ_2, is easily seen to be at a distance R from y given by

$$R = \frac{2R_1 R_2}{R_1 + R_2} = \frac{2}{\frac{1}{R_1} + \frac{1}{R_2}}.$$

Now, if the distance R is the same for each point y of the surface S_y, the surface S_τ generated by the points τ is parallel to the surface S_y. In that case the mean curvature $1/R_1 + 1/R_2$ is also constant. Conversely, if the mean curvature is constant, the surface S_τ is parallel to the surface S_y. We may therefore state the result, which gives a geometric characterization of surfaces of constant mean curvature:

On the normal to a surface S_y at each point y, mark the principal centers of curvature, and let the point τ be the harmonic conjugate of y with respect to these two points. Then the suface S_y is a surface of constant mean curvature if and only if the surface S_τ formed by the points τ is parallel to the surface S_y. In that case the surface S_τ will also be of constant mean curvature.

6. The developables of the axis congruence

To every curve on the surface S_y corresponds a ruled surface of the axis congruence. We proceed now to determine the curves on S_y which correspond to the developables of this congruence. According to the general theory of congruences, there will in general be two one-parameter families of such developables, and consequently a net of curves on S_y corresponding thereto.

As the point y traces out a curve on the surface S_y, the corresponding axis yz generates a ruled surface. The point $dy = y_u\,du + y_v\,dv$ lies on the tangent to the curve traced by the point y, and the point $dz = z_u\,du + z_v\,dv$ lies on the tangent to the curve traced by the corresponding point z. In order that the said ruled surface generated by the line yz be a developable, it is necessary and sufficient that the four points y, z, dy, dz lie in a plane. From equations (65), we find that

$$\begin{aligned} dz &= z_u\,du + z_v\,dv \\ &= [\,(a^{(21)} - a\beta' - \gamma)\,du + (a^{(12)} - a'\gamma - \beta')\,dv\,]\,y_{uv} \\ &\quad + [\,(b^{(21)} - b\beta' - \beta_u)\,du + (b^{(12)} - b'\gamma - \beta'_v)\,dv\,]\,y_u \\ &\quad + [\,(c^{(21)} - c\beta' - \gamma_u)\,du + (c^{(12)} - c'\gamma - \gamma_v)\,dv\,]\,y_v \\ &\quad + [\,(d^{(21)} - d\beta')\,du + (d^{(12)} - d'\gamma)\,dv\,]\,y\,. \end{aligned}$$

We also have

$$dy = y_u\,du + y_v\,dv, \qquad z = y_{uv} - \beta' y_u - \gamma y_v.$$

If, then, the points y, z, dy, dz are to lie in a plane, the determinant of the coefficients of y_{uv}, y_u, y_v which appear in the expressions for z, dy, and dz must vanish. The vanishing of this determinant is moreover sufficient as well as necessary. On expanding the said determinant and equating to zero, a quadratic differential equation is obtained which, if use be made of equations (49), (50), (57), and (59), is without difficulty found to be

$$(74) \qquad Gdu^2 + (F - F' + \gamma_v - \beta'_u)\,dudv - G'\,dv^2 = 0.$$

From the general discussion given at the beginning of Section 2, it follows that the differential equation (74) defines a net of curves on the surface S_y. We shall call this net the *axis net,* and the curves of the net the *axis curves.* Through a point y of the surface pass in general two curves of the net; we shall call the tangents to these curves at the point y the *axis tangents* of the point y.

The quadratic (74) will of course define only a one-parameter family of curves if the discriminant

$$\Delta = (F - F' + \gamma_v - \beta'_u)^2 + 4GG'$$

vanishes. This discriminant is the same as the discriminant of the quadratic (66) which determined the focal sheets of the axis congruence, as was to be expected.

From equation (74) we may read off at once the result which we have already proved geometrically, that the parametric ruled surfaces—i.e., the ruled surfaces of the axis congruence which correspond to the parametric curves C_u and C_v on the surface S_y—are developable if and only if G and G' are zero, in other words, if and only if the curves C_u and C_v are plane curves.

We see also that *the axis tangents corresponding to a point y separate harmonically the parametric tangents at y if and only if*

$$(75) \qquad F - F' + \gamma_v - \beta'_u = 0.$$

If we refer to equations (71) and (72), which define the two covariant points η and η', we find that equation (75) is a necessary and sufficient condition that these two points coincide. We may therefore state the following geometric criterion: *the axis tangents at a point y of the surface separate harmonically the parametric tangents if and only if the two points coincide in which the osculating planes to the curves C_u and C_v at y are tangent to the parametric ruled surfaces $R^{(v)}$ and $R^{(u)}$ respectively.*

Regarding equation (74) as a binary quadratic, and recalling that the asymptotic curves of the surface are given by the quadratic

(18 bis) $$adu^2 + 2dudv + a'\,dv^2 = 0,$$

we may find the condition that the axis curves form a conjugate net, i.e., that the axis tangents separate harmonically the asymptotic tangents. This condition will be the vanishing of the simultaneous invariant (21), which gives

$$a'\,G - aG' - F + F' - \gamma_v + \beta'_u = 0.$$

Using equations (61) and (63), we may state the result in the following form: *the axis curves form a conjugate net if and only if*

(76) $$W^{(v)} - W^{(u)} - \gamma_v + \beta'_u = 0.$$

Let $(dv/du)_1$ and $(dv/du)_2$ denote the roots of equation (74) considered as a quadratic in dv/du. Then the axis tangents of the point y are the lines joining y to the points

$$y_u + \left(\frac{dv}{du}\right)_1 y_v, \qquad y_u + \left(\frac{dv}{du}\right)_2 y_v.$$

Using this fact, we may show without difficulty that *the plane determined by the axis and either of the axis tangents is tangent to each of the parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$ at one of the two focal points.*

This last proposition may also be proved geometrically, if it be remembered that a plane which is tangent to a developable of a congruence is also tangent to one of the focal sheets.

7. The ray congruence and its developables

In Section 3 were introduced the covariant points

$$\rho = y_u - \gamma y, \qquad \sigma = y_v - \beta' y,$$

which we may call respectively the minus first and first Laplace transforms of the point y with respect to the parametric net (C_u, C_v). Let us call the line joining the points ρ and σ the *ray** of the point y; it is a line which lies in the tangent plane to the surface. The rays corresponding to all the points of the surface form a congruence, which we shall call the *ray congruence.**

In general, the ray congruence will have two focal sheets, and on each ray there will be two focal points, i.e., the points at which the ray is tangent to the focal sheets. We may determine the focal points on a ray $\rho\sigma$ as follows. Any point on the ray is given by an expression of the form

(77) $$R = \rho + \mu\sigma.$$

* The names *ray* and *ray congruence* were applied by Wilczynski to the corresponding case in which the parametric net is conjugate. See his memoir, *The general theory of congruences*, these Transactions, vol. 16 (1915), pp. 311–327.

The totality of points R corresponding to the points y of the surfaces S_y form a surface S_R. We wish to determine μ as a function of u, v so that the surface S_R shall be a focal sheet, i.e., so that the line $\rho\sigma$ shall be tangent to the surface S_R. In other words, the points ρ, σ, R_u, R_v are to lie in a plane. We have

$$(78) \qquad \begin{aligned} \rho_u &= ay_{uv} + (b - \gamma) y_u + cy_v + (d - \gamma_u) y, \\ \rho_v &= y_{uv} - \gamma y_v - \gamma_v y, \\ \sigma_u &= y_{uv} - \beta' y_u - \beta'_u y, \\ \sigma_v &= a' y_{uv} + b' y_u + (c' - \beta') y_v + (d' - \beta'_v) y, \end{aligned}$$

so that

$$\begin{aligned} R_u - \mu_u \sigma &= \rho_u + \mu\sigma_u \\ &= (a + \mu) y_{uv} + (b - \gamma - \beta' \mu) y_u + cy_v + (d - \gamma_u - \mu\beta'_u) y, \\ R_v - \mu_v \sigma &= \rho_v + \mu\sigma_v \\ &= (1 + a' \mu) y_{uv} + b' \mu y_u + [(c' - \beta') \mu - \gamma] y_v \\ &\qquad + [(d' - \beta'_v) \mu - \gamma_v] y. \end{aligned}$$

Also,

$$\rho = y_u - \gamma y, \qquad \sigma = y_v - \beta' y.$$

If, now, the points ρ, σ, R_u, R_v are to lie in a plane, it is necessary and sufficient that in the expressions just written for ρ, σ, $R_u - \mu_u \sigma$, $R_v - \mu_v \sigma$ the determinant of the coefficients of y_{uv}, y_u, y_v, y vanish. This yields a quadratic equation for μ, which is at once found to be

$$(79) \qquad h' \mu^2 + [ah' - a' k + (1 - aa')(\beta'_u - \gamma_v)] \mu - k = 0,$$

where h' and k are invariants given by equations (49) and (50). *The focal sheets of the ray congruence are given by the covariants*

$$(80) \qquad R_1 = \rho + \mu_1 \sigma, \qquad R_2 = \rho + \mu_2 \sigma,$$

where μ_1 and μ_2 are the two roots of the quadratic (79).

We may write the quadratic (79) in two other forms which are sometimes more useful. From (58) and (61) we find that

$$(81) \qquad k = (1 - aa') G - aW^{(u)}, \qquad a' k = (1 - aa') F - W^{(u)}.$$

Similarly, we have the relations

$$(82) \qquad h' = (1 - aa') G' - a' W^{(v)}, \qquad ah' = (1 - aa') F' - W^{(v)}.$$

Equation (79) may therefore be written in either of the following ways:

$$(83) \qquad h' \mu^2 + [W^{(u)} - W^{(v)} + (1 - aa')(F' - F + \beta'_u - \gamma_v)] \mu - k = 0,$$

$$\left(G' - \frac{a' W^{(v)}}{1 - aa'}\right)\mu^2 + \left[F' - F + \beta'_u - \gamma_v + \frac{W^{(u)} - W^{(v)}}{1 - aa'}\right]\mu - \left(G - \frac{aW^{(u)}}{1 - aa'}\right) = 0. \tag{84}$$

We recall that the axis curves, i.e., the curves in which the developables of the axis congruence cut the surface S_y, are defined by the differential equation

$$Gdu^2 + (F - F' + \gamma_v - \beta'_u)\,dudv - G'\,dv^2 = 0. \tag{74 bis}$$

Consequently the axis tangents of a point y of the surface are the lines joining that point with the points

$$\rho + \left(\frac{dv}{du}\right)_1 \sigma, \qquad \rho + \left(\frac{dv}{du}\right)_2 \sigma,$$

where $(dv/du)_1$ and $(dv/du)_2$ are the roots of the quadratic (74). Using (84) as the equation for the determination of the μ_1 and μ_2 of (80), we see that the points R_1 and R_2, the focal points of the ray, lie on the axis tangents if and only if $W^{(u)}$ and $W^{(v)}$ are both zero. *The axis tangents at each point of the surface S_y intersect the corresponding ray in the focal points of the ray if and only if both of the congruences of tangents to the parametric curves C_u and C_v on S_y are W-congruences.*

We proceed now to set up the differential equation which defines the developables of the ray congruence. As the point y traces out a curve on the surface S_y, the corresponding rays $\rho\sigma$ generate a ruled surface of the ray congruence. We seek those curves on S_y for which the corresponding ruled surfaces of the ray congruence are developable. We shall call these curves the *ray curves* of the surface. They are of course determined by the particular parametric net to which the surface is referred.

If the line $\rho\sigma$ is to generate a developable, the four points ρ, σ, $d\rho = \rho_u\,du + \rho_v\,dv$, $d\sigma = \sigma_u\,du + \sigma_v\,dv$ must lie in a plane. Using equations (78), we find immediately that

$$\begin{aligned} d\rho &= (adu + dv)\,y_{uv} + (b - \gamma)\,du \cdot y_u + (cdu - \gamma dv)\,y_v \\ &\qquad + [(d - \gamma_u)\,du - \gamma_v\,dv]\,y, \\ d\sigma &= (du + a'\,dv)\,y_{uv} + (-\beta'\,du + b'\,dv)\,y_u + (c' - \beta')\,dv \cdot y_v \\ &\qquad + [-\beta'_u\,du + (d' - \beta'_v)\,dv]\,y. \end{aligned}$$

Equating to zero the determinant formed from the coefficients of y_{uv}, y_u, y_v, y in the expressions for $d\rho$, $d\sigma$, and $\rho = y_u - \gamma y$, $\sigma = y_v - \beta' y$, we obtain the condition that these last four points lie in a plane. On expanding the determinant, it is found without difficulty that *the ray curves of the surface S_y are defined by the differential equation*

$$
\begin{aligned}
(85)\qquad & [k + a(\beta'_u - \gamma_v)]\,du^2 + [a'k - ah' + (1 + aa')(\beta'_u - \gamma_v)]\,dudv \\
& \qquad - [h' - a'(\beta'_u - \gamma_v)]\,dv^2 = 0,
\end{aligned}
$$

where the Laplace-Darboux invariants k and h' are given by equations (49) and (50).

As was to be expected, the ray curves form in general a net of curves on the surface S_y. We shall call the two tangents at a point y to the two curves of the net which pass through y, the *ray tangents* of the point y.

A comparison of equations (79) and (85) will show at once that *the ray tangents meet the ray in the focal points of the ray if and only if*

$$
(86)\qquad \beta'_u - \gamma_v = 0.
$$

We recall once more that the asymptotic curves of the surface S_y are defined by the quadratic (18),

$$
adu^2 + 2dudv + a'dv^2 = 0.
$$

The harmonic invariant of this and (85) is $(1 - aa')(\beta'_u - \gamma_v)$. This can vanish if and only if $\beta'_u - \gamma_v = 0$, since we have supposed $1 - aa'$ to be nowhere zero. Combining this with the result previously obtained, we see that *the ray curves form a conjugate net if and only if* $\beta'_u - \gamma_v = 0$, *that is, if and only if at each point of the surface the ray tangents meet the corresponding ray in the focal points of the ray.*

An interesting special case of this last theorem is that in which the axis curves also form a conjugate net. In that case equation (76) is satisfied in addition to equation (86). *The ray curves and axis curves both form conjugate nets if and only if*

$$
(87)\qquad \beta'_u - \gamma_v = 0, \qquad W^{(u)} - W^{(v)} = 0.
$$

In virtue of equations (81) and (82), the differential equation of the axis curves, (74), may be written

$$
\begin{aligned}
(88)\qquad & (k + aW^{(u)})\,du^2 + [a'k - ah' + W^{(u)} - W^{(v)} \\
& \qquad - (1 - aa')(\beta'_u - \gamma_v)]\,dudv - (h' + a'W^{(v)})\,dv^2 = 0
\end{aligned}
$$

Comparing this with equation (85), we observe that *the axis curves coincide with the ray curves if and only if*

$$
(89)\qquad W^{(u)} - (\beta'_u - \gamma_v) = 0, \qquad W^{(v)} + (\beta'_u - \gamma_v) = 0.
$$

We may use the results just obtained in stating the following theorem.

THEOREM. *If for the parametric net on the surface S_y the developables of the axis and ray congruences correspond, and if the axis and ray curves both form conjugate nets, then the congruences of tangents to the two parametric families C_u and C_v are W-congruences.*

Another geometric criterion that the ray curves form a conjugate net may be obtained as follows. From the second of equations (78), we find at once that

$$\begin{aligned} \rho_v - \beta'\rho &= y_{uv} - \beta' y_u - \gamma y_v + (\beta'\gamma - \gamma_v)y \\ &= z + (\beta'\gamma - \gamma_v)y. \end{aligned} \tag{90}$$

Now, the line $\rho\rho_v$ is the tangent at ρ to the curve $u =$ const. on the surface S_ρ. The left-hand member of equation (90) is a point on this tangent, and the right-hand member is a point on the axis of the point y. Therefore, *the axis yz of a point on the surface S_y is met in the point*

$$\zeta = z + (\beta'\gamma - \gamma_v)y \tag{91}$$

by the line $\rho\rho_v$, which is tangent to the corresponding curve $u =$ const. on the surface S_ρ.

Similarly, *the tangent $\sigma\sigma_u$ to a curve $v =$ const. on the surface S_σ intersects the corresponding axis yz in the point*

$$\zeta' = z + (\beta'\gamma - \beta'_u)y. \tag{92}$$

Consequently, *the points ζ and ζ' coincide if and only if $\beta'_u = \gamma_v$, that is, if and only if the ray curves form a conjugate net.*

In Section 5, we found that the osculating plane to the curve C_v at the point y is tangent at the point

$$\eta = z - (F' - \beta'\gamma + \beta'_u)y \tag{71 bis}$$

to the ruled surface $R^{(u)}$ of the axis congruence which corresponds to the curve C_u. The point η coincides with the point ζ' given by (92) if and only if $F' = 0$. But the vanishing of the invariant F', which is given by equation (59), is a necessary and sufficient condition that the curves $v =$ const. on S_σ be asymptotics. Hence, *the points η coincide with the points ζ' if and only if the curve $v =$ const. on the surface S_σ are asymptotic.*

Similarly, *the points η' given by equation* (72) *coincide with the points ζ if and only if the curves $u =$ const. on the surface S_ρ are asymptotic.*